Finding a colour with AA contrast ratio

The WCAG AA contrast requirement for text requires a contrast ratio between the text and the background of at least 4.5:1. Given a background colour, suppose we want to find a colour that satisfies this contrast ratio as quickly as possible.

The two most opposite colours are white (`#ffffff`) and black (`#000000`), so it makes sense to try those first.

Claim. *Every colour has a contrast ratio of at least 4.5:1 with at least one of white or black.*

Proof. The WCAG defines the *contrast ratio* of two colours to be

$$\frac{L_1 + 0.05}{L_2 + 0.05},$$

where L_1 is the relative luminance of the lighter colour, and L_2 is the relative luminance of the darker colour. (See WCAG technique G17.)

The relative luminance of white is $L_{\text{white}} = 1$, and the relative luminance of black is $L_{\text{black}} = 0$.

Suppose there were a colour with relative luminance L, which has insufficient contrast with both white and black. It must satisfy both:

$$\frac{L_{\text{white}} + 0.05}{L + 0.05} < 4.5 \qquad \text{and} \qquad \frac{L + 0.05}{L_{\text{black}} + 0.05} < 4.5$$

The luminance of a colour always satisfies $0 \leq L \leq 1$, so we can simplify these to:

$$L > \frac{11}{60} = 0.18333\ldots \qquad \text{and} \qquad L < \frac{7}{40} = 0.175.$$

This is a contradiction, which means there is no colour which has insufficient contrast with both white and black.

This means we can always find a colour with sufficient contrast in at most two lookups: first we try white, then we try black. □

Finding a colour with AAA contrast ratio

The enhanced contrast requirement requires a contrast ratio between the text and the background of at least 7. Can we use white and black to find a guaranteed colour with this contrast ratio?

It turns out not: if you try to repeat the proof above with a contrast ratio of 7, not 4.5, you don't get a contradiction. Instead, yopu learn that a colour has insufficient contrast with both white and black if and only if

$$0.1 < L < 0.3$$

and there are colours with this luminance.

If you work through the greys, you find `#5a5a5a`, which has a relative luminance of 0.102, a contrast ratio 6.897:1 with white and 3.045:1 with black. If you go right to the middle, `#7f7f7f` has a contrast ratio 4.004:1 with white and 5.245:1 with black.

Indeed, there are no colours that have a contrast ratio of 7:1 with `#7f7f7f`. (For proof, imagine such a colour with luminance L, and consider the cases $L < L_{\texttt{\#7f7f7f}}$ and $L > L_{\texttt{\#7f7f7f}}$. In both cases you discover the contrast is less than 7.) As you keep increasing your contrast requirement, some colours become unusable. Eventually you reach the maximum contrast ratio of 21:1, when the only colours you can use are black and white.